EXERCISE SHEET 1

Analysis II-MATH-106 (en) EPFL
Spring Semester 2024-2025
February 17, 2025

Exercise 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable on $\mathbb{R}$ with $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Indicate for each of the following statement if it is True or False:

(a) If $f(a) = g(a) = 0$ for $a \in \mathbb{R}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.
(b) If $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$, then $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$.
(c) If $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$ does not exist, then $\lim_{x\to+\infty} \frac{f(x)}{g(x)}$ does not exist.
(d) If there exist $x \neq y \in \mathbb{R}$ such that $f(y) - f(x) = g(y) - g(x)$, then there exists $c \in]x, y[$ such that $f'(c) = g'(c)$.
(e) Let $a \in \mathbb{R}$, then $\lim_{x\to a} \frac{\sin g(x)}{g(x)}$ exists.
(f) Let $a \in \mathbb{R}$, then $\lim_{x\to a} \frac{\sinh g(x)}{g(x)} = \cosh g(a)$.

Exercise 2. Let I be an open interval, $f, g \in C^n(I)$ and $a \in I$. Indicate for each of the following statement if it is True or False:

(a) If $f^{(k)}(a) = 0$ for all $0 \leq k < 7$ and $f^{(7)}(a) = 1$, then f has a minimum at a.
(b) If I is symmetric and f is odd on I, then $f^{(2k)}(0) = 0$ for $0 \leq 2k \leq n, k \in \mathbb{N}$.
(c) If $f^{(k)}(a) = g^{(k)}(a) = 0$ for all $0 \leq k < n$ and $g^{(n)}(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$.

Exercise 3. Calculate the following limits:

(a)
$$\lim_{x\to 0} \frac{\sin x}{\ln\left(\frac{1}{1+x}\right)},$$

(b)
$$\lim_{x\to 0+} x^\alpha \ln x, \quad \alpha > 0,$$

(c)
$$\lim_{x\to 1} \frac{\arctan\left(\frac{1-x}{1+x}\right)}{x-1},$$

(d)
$$\lim_{x\to 0} \frac{\ln\left(1+x^2\right)}{\sinh^2 x},$$

(e)

$$\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)\sin(x-1)}{\ln\left((x-1)^2\right)},$$

(f)

$$\lim_{x\to 0+} \frac{\cos x - \cos\frac{1}{x}}{e^x - e^{\frac{1}{x}}}.$$

Exercise 4. For each of the following functions, find the stationary points, local and global extrema and inflexion points (if any), the intervals of increase and decrease, and describe the asymptotic behaviour (as x approaches $\pm\infty$ and at the boundary of the domain of definition, if there is one).

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = xe^{-x^2}$.
(b) $f :]-\infty, -1] \cup [3, \infty[\to \mathbb{R}$ defined by $f(x) = \sqrt{x^2 - 2x - 3}$.

Exercise 5. Let $f :]0, \infty[\longrightarrow \mathbb{R}$ be a function defined by $f(x) = x^x e^{-x}$.

(a) Show that $x = 1$ is the unique stationary point of the function f.
(b) Study its nature and give the truncated expansion of order 4 at this point.
(c) Show that $\lim_{x\to 0_+} f(x) = 1$ and calculate

$$\lim_{x\to 0_+} \frac{f(x) - 1}{x \ln x}.$$

Exercise 6. Compute the following integrals:

(a)

$$\int_0^{+\infty} e^{-\sqrt{t}}\, \mathrm{d}t,$$

(b)

$$\int_1^{+\infty} \frac{\ln t}{t^3}\, \mathrm{d}t,$$

(c)

$$\int_0^{+\infty} \frac{\arctan t}{1+t^2}\, \mathrm{d}t.$$

Exercise 7. Determine, in terms of the real number $\alpha > 0$, the convergence of the following generalized integrals:

(a)

$$\int_0^1 \frac{1}{t^\alpha}\, \mathrm{d}t,$$

(b)

$$\int_1^{\infty} \frac{\ln t}{t^\alpha}\, \mathrm{d}t.$$